Least Squares

- Given N equations in N unknowns, one can usually solve them for a unique answer. Thus, given

  $$y = f(a)$$

  with y and a each a vector with N components, we can usually solve for *a* given *y*.

Least Squares (2)

- However, if the y's are only known approximately, e.g.,

  $$y = \hat{y} + \varepsilon = f(a),$$

  say, with ε an error vector and the 'hats' indicating the observed values, then when we solve the equation

  $$\hat{y} = f(\hat{a}),$$

  we will make an error, since the a's that we calculate will differ from the true values by some small unknown amount:

  $$a = \hat{a} + \delta$$

Least Squares (3)

- If we have M>N equations (so that there are more equations than there are unknowns) there is in general no solution at all.
- However, since we now have more information about the observed y's, we ought to be able to get better information about a if we can invent a suitable strategy for solving

  $$\hat{y} \cong f(\hat{a})$$

  even if inexactly.

Least Squares (4)

- The idea of least squares is to find the value of $\hat{a}$ that makes

  $$\sum \varepsilon^2 = \sum (f(\hat{a}) - \hat{y})^2$$

  as small as possible.
- This is a problem that is readily solved by calculus (find the partial derivatives with respect to all of the components of $\hat{a}$ and set them to zero).

Example

- Suppose we observed at least 4 stars and measured their (x,y) positions on the CCD
- Suppose we calculate the standard coordinates of the stars (X,Y) from a catalog of known positions
- Assume a six-constant plate model (a,b,c,d,e,f) (so there are 6 unknowns and 8 measurements...the system is overdetermined and we can use Least Squares).
- Write:

$$X_i = ax_i + by_i + c \; (4\, equations\, in\, 3\, unknowns),$$
$$Y_i = dx_i + ey_i + f \; (4\, equations\, in\, 3\, unknowns)$$

Example (2)

- We wish to minimize $S^2 = S_x^2 + S_y^2$, where

$$S_x^2 = \sum_i (X_i - ax_i - by_i - c)^2,$$
$$S_y^2 = \sum_i (Y_i - dx_i - ey_i - f)^2$$

by choosing (a,b,c,d,e,f).
- The quantities S_x^2 and S_y^2 are *figures of merit* that tell us how well we were able to fit the data.
- From them we can derive information that tells us how good the approximations (a,b,c,d,e,f) are.

Example (3)

- After obtaining (a,b,c,d,e,f), we can then compute the X and Y for any other star on the CCD image (not in the catalog) by measuring the (x,y) of the star and plugging into

$$X = ax + by + c,$$
$$Y = dx + ey + f$$

to get (X,Y). Then we can determine the corresponding (α,δ) using the inverse equations for standard coordinates to equatorial coordinates.

Example (4)

- For simplicity, let's just consider the problem of minimizing just with respect to (a,b,c).
- Taking partial derivatives with respect to (a,b,c), we find that

$$-\frac{1}{2}\frac{\partial S^2}{\partial a} = \sum_i (X_i x_i - ax_i^2 - bx_i y_i - cx_i) = 0,$$
$$-\frac{1}{2}\frac{\partial S^2}{\partial b} = \sum_i (X_i y_i - ax_i y_i - by_i^2 - cy_i) = 0,$$
$$-\frac{1}{2}\frac{\partial S^2}{\partial c} = \sum_i (X_i - ax_i - by_i - c) = 0,$$

- We get three similar but independent equations for (d,e,f).

Normal Equations

- These are the *normal equations*. They are linear in (a,b,c) and can be solved in the usual way. A more convenient way to write them is in matrix form:

$$\begin{bmatrix} \sum_i x_i^2 & \sum_i x_i y_i & \sum_i x_i \\ \sum_i x_i y_i & \sum_i y_i^2 & \sum_i y_i \\ \sum_i x_i & \sum_i y_i & N \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_i X_i x_i \\ \sum_i X_i y_i \\ \sum_i X_i \end{bmatrix}$$

$$\mathbf{Na} = \mathbf{x},\ \mathbf{a} = \mathbf{N}^{-1}\mathbf{x}$$

Normal Equations (2)

- In this case, the normal equations for (a,b,c,d,e,f) partition so that the (a,b,c) solution is independent of the (d,e,f) solution. This is not usually the case

$$\left[\begin{array}{ccc|ccc} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{array}\right] \left[\begin{array}{c} a \\ b \\ c \\ \hline d \\ e \\ f \end{array}\right] = \left[\begin{array}{c} * \\ * \\ * \\ \hline * \\ * \\ * \end{array}\right]$$

Example

- Suppose we've observed 4 standard stars. We know the right ascensions and declinations from a catalog, and we've computed the standard coordinates (X,Y) assuming a tangential point at a particular (A,D)=(32°,46°).
- Summarizing, we get the information in the following table. Compute (a,b,c) for the six plate constant model

Star	α	δ	X	x	y
1	30°	45°	–0.02469	0	100
2	31°	44°	–0.01256	100	0
3	30.75°	45°	–0.01543	75	100
4	30.25°	44.333°	–0.02186	25	30

Example (2)

- The normal equations for (a,b,c) are (in matrix form)

$$\begin{bmatrix} 16250 & 8250 & 200 \\ 8250 & 20900 & 230 \\ 200 & 230 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2.9600 \\ -4.6676 \\ -0.0745 \end{bmatrix}$$

Example (3)

- Using a spreadsheet to invert the matrix and multiply, I get the result:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.00020518 & 8.6884\times10^{-5} & -0.0152548 \\ 8.6884\times10^{-5} & 0.00016708 & -0.0139516 \\ -0.0152548 & -0.0139516 & 1.8149541 \end{bmatrix} \begin{bmatrix} -2.9600 \\ -4.6676 \\ -0.0745 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00012422 \\ 2.8869\times10^{-6} \\ -0.02501204 \end{bmatrix}$$

Example (4)

- Suppose we had a star whose position we needed, but measured at (x,y)=(45,65). What is the star's standard coordinate X? We get:

$$X = 45\times0.00012422 + 65\times2.8869\times10^{-6} - 0.02501204$$
$$= -0.019234$$

- If we also had the star's Y value, we would also be able to compute its right ascension and declination (this requires a solution for the (d,e,f) constants).

General Case

- In general, one cannot separate the X and Y solutions as we have done. This is because in the general case, some constants may appear in both solutions. For example, in the four plate constant model, (a,b) appear in both the X and Y equations. Or, in the model with tilt terms, the constants (p,q) appear in both X and Y equations.
- As a consequence, we usually have to make a solution that involves all of the (X,Y) observations simultaneously.